OLG

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This course provides an overview to the Overlapping Generations (OLG) model, one of the most versatile and influential frameworks in modern macroeconomics. Unlike the Ramsey model with infinitely-lived agents, the OLG framework captures the realistic life-cycle dynamics where individuals live for finite periods, making decisions about consumption, savings, and intergenerational transfers.

The OLG model is particularly powerful for analyzing issues that involve intergenerational considerations, such as social security systems, public debt sustainability, educational policies, and long-run economic development. The course combines theoretical foundations with applications to contemporary economic research, demonstrating how this framework helps us understand fundamental questions about economic growth, inequality, and policy design.

Through the study of cutting-edge research papers, we explore how economists use the OLG framework to investigate diverse phenomena from the evolution of social preferences to environmental sustainability.

2.1 Course Objectives

Upon completion of this course, students will be able to:

Theoretical Foundations: - Understand the core OLG model with its key assumptions - Use the OLG model to discuss about different issues

Technical Skills: - Solve household optimization problems in a life-cycle context with perfect foresight

Research Applications: - Critically analyze how contemporary researchers extend the basic OLG framework - Understand applications to cultural evolution, environmental economics, and development - Develop skills to formulate and solve original research questions using the OLG framework

2.2 Prerequisites

- Intermediate Microeconomics (consumer theory, optimization)
- Mathematical Methods for Economics (dynamic systems)
- Basic knowledge of econometrics is helpful but not required

2.3 Course Format

- Duration: 10 sessions (2 hours per session)

2.4 Assessment

Student evaluation consists of:

- **Written Report with Oral Defense (100%)**: Comment and discuss one of the proposed articles using the OLG model.
  - The objective of the report is to demonstrate your understanding of the **core** ideas of the article, **not every technical detail**.
  - The report shall discuss:
    - What the article is about and why it is important

    * Description of the model (general structure, key assumptions, preferences, technologies, equilibrium, dynamics)
    * Key results and economic intuition
    * Critical assessment (strengths, weaknesses, possible extensions)

2.5 Contact Information

- Instructor: Eric Roca (eric.roca_fernandez@uca.fr)
- Office Hours: By appointment

2.6 Bibliography

2.6.1 Core Textbooks

- Croix and Michel (2009)
- Campante, Sturzenegger, and Velasco (2021) Freely available online

2.6.2 Research Papers

- Diamond (1965)
- Galor and Moav (2006)
- Galor and Özak (2016)
- Croix and Dottori (2008)

3 The OLG model

3.1 Introduction

Based on Croix and Michel (2009).

The overlapping generations (OLG) model focuses on the life-cycle: agents make decisions regarding how to consume and how much to save for retirement. Individuals live for two periods of time, work when young, and retire when old. This framework is particularly useful for studying intergenerational redistribution, allowing us to analyze:

- social security systems,
- education policies, and
- public debt.

The key feature of the OLG model is that agents are heterogeneous by age. In the first period of life, individuals are young and work. When old, they retire and live from savings. Hence, at any point in time, *two types of agents with* ***different*** *budget constraints coexist*: young and old.[1]

A basic reference for this model is Diamond (1965).

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

3.2 Preliminaries

In this model, time is discrete and extends from $t = 0, 1, \dots, \infty$. Individuals make decisions at points in time. We shall have initial conditions detailing the state of the economy at $t = 0$.

3.2.1 Individuals live for two periods

Individuals live for two periods, meaning that, at every point in time, two generations are alive and overlap.

This is relevant: the economy goes on forever but individuals only operate during some periods. Hence, there will be infinite two-period-lived generations. In particular, at $t = 0$, we will have a young and an adult generation. This adult generation will die at the end of $t = 1$, the young generation will become adults and have children: the *new* young generation of $t = 2$. Hence, we can represent the generations diagrammatically —in brackets I have denoted the year in which each generation was born.

$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
Old (t=-1)	Die				
Young(t=0)	Old (t=0)	Die			
Young(t=1)	Old (t=1)	Die			
	Young(t=2)	Old (t=2)	Die		
		Young(t=3)	Old (t=3)	Die	
			Young(t=4)	Old (t=4)	
				Young(t=5)	

To simplify the model, we assume that each adult born in $t-1$ has $n > -1$ children.

Note: we assume the number of children to be constant. More complex set-ups include endogenous fertility (see Section 3.9).

These are the young population at time t Therefore, the total population N at time t is composed of adults and young people.

$$N_t = \underbrace{N_{t-1}}_{\text{Adults}} + \underbrace{N_{t-1}n}_{\text{Youngs}} = N_{t-1}(1+n).$$

The total population at any time t is:

$$N_t = N_0(1+n)^t.$$

3.3 Assumptions

3.3.1 Firms

We assume that a large number of identical firms populate the economy. Firms produce a single, homogeneous good using capital and labour. Each firm solves the following profit maximization problem:

$$\max_{K_t, L_t} \quad F(K_t, L_t) - r_t K_t - w_t L_t$$

where $F(K,L)$ is the production function, r_t is the rental rate of capital, and w_t is the wage rate.

We assume the production function satisfies standard neoclassical properties:

Assumption OLG 1: The production function $F(K,L)$ is:

- **OLG 1.1** Continuous and twice continuously differentiable,

- **OLG 1.2** Strictly increasing in both arguments: $F_K(K, L) > 0$ and $F_L(K, L) > 0$,
- **OLG 1.3** Strictly concave: $F_{KK}(K, L) < 0$, $F_{LL}(K, L) < 0$,
- **OLG 1.4** Homogeneous of degree one (constant returns to scale),
- **OLG 1.5** Satisfies the Inada conditions:

$$\lim_{K \to 0} F_K(K, L) = \lim_{L \to 0} F_L(K, L) = +\infty$$
$$\lim_{K \to +\infty} F_K(K, L) = \lim_{L \to +\infty} F_L(K, L) = 0.$$

Since there are many firms competing in perfect competition, in equilibrium firms make zero profits and factors are paid their marginal products. The first-order conditions for profit maximization are:

$$F_K(K_t, L_t) = r_t$$
$$F_L(K_t, L_t) = w_t$$

Since the production function F is homogeneous of degree one, we can write it in *intensive* (per-worker) terms:

$$f(k) \equiv F\left(\frac{K}{L}, 1\right), \quad k \equiv \frac{K}{L}. \tag{3.1}$$

Using Euler's theorem for homogeneous functions, we have $F(K, L) = F_K(K, L) \cdot K + F_L(K, L) \cdot L$. Dividing by L and using the definition of k:

$$f(k) = f'(k) \cdot k + w$$

Therefore, the factor prices in intensive form are:

$$r_t = f'(k_t) \tag{3.2}$$

$$w_t = f(k_t) - f'(k_t)k_t. \tag{3.3}$$

3.3.1.1 Example: Cobb-Douglas production function

We now derive the key relationships for the Cobb-Douglas production function, which we will use throughout the rest of the chapter. Consider:

$$F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}, \quad \alpha \in (0, 1).$$

This function satisfies all our assumptions (OLG 1.1–1.5). In intensive form:

$$f(k_t) = F\left(\frac{K_t}{L_t}, 1\right) = \left(\frac{K_t}{L_t}\right)^\alpha = k_t^\alpha.$$

The marginal products are:

$$\begin{aligned} f'(k_t) &= \alpha k_t^{\alpha-1} \\ f(k_t) - f'(k_t)k_t &= k_t^\alpha - \alpha k_t^{\alpha-1} \cdot k_t = (1-\alpha)k_t^\alpha \end{aligned}$$

Therefore, under Cobb-Douglas production, the factor prices are:

$$\begin{aligned} r_t &= \alpha k_t^{\alpha-1} \\ w_t &= (1-\alpha)k_t^\alpha \end{aligned}$$

{#eq-cobb_douglas_prices}

Note: The parameter α represents capital's share in total output, while $(1-\alpha)$ is labour's share. These shares remain constant regardless of the capital-labour ratio k_t.

3.3.2 Households

Individuals live for two periods. As before, we assume perfect foresight for individuals.

Assumption OLG 2 Individuals have perfect foresight.

When **young**, they are endowed with *one unit of labour* that they *supply inestastically.*

Assumption OLG 3 Individuals supply one unit of labour inelastically when young. They receive the ongoing wage rate w_t and allocate this income between:

- current consumption c_t,
- savings s_t that are invested in the firms.

Therefore, the budget constraint of a *young* individual in period t is:

$$w_t = c_t + s_t.$$

Once an individual reaches old age the next period, he consumes his savings (plus the interest rate received), reproduces —exogenous fertility at rate n— and dies. Old people *do not care* about anything happening after death. Therefore, an agent has one unique choice:

- consumption when adult, d_{t+1}.

The budget constraint for this period is:

$$s_t(1 - \delta + r_{t+1}) = d_{t+1}.$$

with $\delta \in (0, 1)$ being the capital depreciation rate.

Hence, an individual faces two budget constraints. However, we can collapse both into a unique *intertemporal budget constraint.*

3.3.2.1 The intertemporal budget constraint

In the economy, we have consumption as the numeraire. It is more convenient for us to combine the two budget constraints corresponding to young and old ages into one single constraint. Starting from

$$\begin{cases} w_t = c_t + s_t \\ d_{t+1} = s_t(1 - \delta + r_{t+1}) = s_t R_{t+1} \end{cases} \tag{3.4}$$

where $R_t \equiv 1 - \delta + r_{t+1}$ represents the return on savings, isolate s_t in the second equation and plug it in the first one:

$$w_t = c_t + \frac{d_{t+1}}{R_{t+1}}. \tag{3.5}$$

The intertemporal budget constraint indicates that the total present value of income (w_t, the only source of income) equals the total present value of expenditures. The present value of consumption when old d_{t+1} is discounted using the interest rate R_{t+1}.

It is clear that savings, as usual, will be a function of wages w and interests R. So will consumption at all periods of time.

3.3.2.2 Utility function

We suppose that the life-cycle utility function is additively separable:

$$U(c,d) = u(c) + \beta u(d),\ \beta \in (0,1) \tag{3.6}$$

where $\beta \in (0,1)$ is the psychological discount factor. We assume that $u(c)$ has the properties

Assumption OLG 4

- **OLG 4.1** $u'(c) > 0$,
- **OLG 4.2** $u''(c) < 0$,
- **OLG 4.3** $\lim_{c\to 0} u'(c) = +\infty$.

The last assumption $\lim_{c\to 0} u'(c) = +\infty$ implies that an individual will always have a positive consumption —as long as he has enough income to finance it.

Another important implication of the choice of the utility formulation is that c and d are normal goods: the demand is *not decreasing* in wealth. It follows from additive separability and concavity.

3.3.3 The behaviour of individuals

At time t, young individuals receive their wages, consume and save while maximising the utility function.

$$\begin{aligned} &\max u(c_t) + \beta u(d_{t+1}) \\ \text{s.t.} \quad & w_t = c_t + s_t \\ & d_{t+1} = R_{t+1} s_t \\ & c_t \geq 0, d_{t+1} \geq 0. \end{aligned}$$

{#eq-optimization_problem}

We have two possibilities to solve this problem:

3.3.3.1 Substitution

First, we can substitute c_t and d_{t+1} in the utility function, leading to:

$$u(w_t - s_t) + \beta u(R_{t+1} s_t).$$

This function is strictly concave with respect to s_t because of our assumptions. The solution is *the savings function*:

$$s_t = s(w_t, R_{t+1}).$$

The solution is interior as a consequence of the assumptions, and it is characterised by the first-order condition:

$$u'(w_t - s_t) = \beta R_{t+1} u'(R_{t+1} s_t). \tag{3.7}$$

3.3.3.2 Lagrangian

Instead, we can use the intertemporal budget constraint and build the Lagrangian:

$$\mathcal{L} = u(c_t) + \beta u(d_{t+1}) + \lambda_t (w_t - c_t - \frac{d_{t+1}}{R_{t+1}}).$$

The first order conditions imply that:

$$u'(c_t) = \lambda_t, \quad \beta u'(d_{t+1}) = \frac{\lambda_t}{R_{t+1}}.$$

Combining both, we obtain Equation 3.7 again:

$$u'(c_t) = \beta R_{t+1} u'(d_{t+1}).$$

3.4 Examples of utility functions and savings behavior

To understand how individuals save, we now consider two specific cases that illustrate different aspects of the savings decision.

3.4.1 Example 1: Log-utility

The simplest case is logarithmic utility:

$$u(c) = \log(c).$$

This is a special case of the CIES utility function with $\sigma = 1$. From the Euler equation Equation 3.7, we have:

$$\frac{1}{w_t - s_t} = \beta R_{t+1} \frac{1}{R_{t+1} s_t}$$

Simplifying:

$$\frac{1}{w_t - s_t} = \frac{\beta}{s_t}$$

Solving for s_t:

$$s_t = \beta(w_t - s_t) \implies s_t(1+\beta) = \beta w_t$$

Therefore, the savings function under log-utility is:

$$s_t = \frac{\beta}{1+\beta} w_t. \tag{3.8}$$

Key properties:

- Savings are a constant fraction of wage income,
- Savings are **independent** of the interest rate R_{t+1},
- The marginal propensity to save is $\frac{\beta}{1+\beta} \in (0,1)$.

This result shows that under log-utility, the wealth effect and substitution effect of interest rate changes exactly cancel out.

3.4.2 Example 2: General CIES utility

Now consider the constant intertemporal elasticity of substitution (CIES) utility function:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \sigma > 0.$$

The parameter σ is the **intertemporal elasticity of substitution**: it measures the willingness to substitute consumption across time in response to changes in relative prices (the interest rate).

From the Euler equation:

$$(w_t - s_t)^{-\frac{1}{\sigma}} = \beta R_{t+1}(R_{t+1} s_t)^{-\frac{1}{\sigma}}$$

Rearranging:

$$\frac{d_{t+1}}{c_t} = \frac{R_{t+1}s_t}{w_t - s_t} = (\beta R_{t+1})^{\sigma}. \tag{3.9}$$

While we cannot solve explicitly for s_t in general, we can analyze how savings respond to changes in the interest rate using **implicit differentiation**.

3.4.2.1 The effect of interest rates on savings: implicit differentiation

To analyze how savings respond to changes in the interest rate, we use the **implicit function theorem**. Define the function $\phi(s, w, R)$ from the Euler equation:

$$\phi(s, w, R) = -u'(w - s) + \beta R u'(Rs) = 0.$$

This implicitly defines the savings function $s = s(w, R)$. By the implicit function theorem:

$$\frac{\partial s}{\partial R} = -\frac{\frac{\partial \phi}{\partial R}}{\frac{\partial \phi}{\partial s}}.$$

Computing the partial derivatives:

$$\begin{aligned}\frac{\partial \phi}{\partial R} &= \beta u'(Rs) + \beta R u''(Rs)s = \beta u'(d)\left[1 + \frac{u''(d)}{u'(d)}Rs\right] \\ &= \beta u'(d)\left[1 - \frac{1}{\sigma(d)}\right]\end{aligned}$$

where we use the fact that $\sigma(d) = -\frac{u'(d)}{u''(d)d}$ is the elasticity of intertemporal substitution evaluated at $d = Rs$.

Similarly:

$$\frac{\partial \phi}{\partial s} = u''(w-s) + \beta R^2 u''(Rs) < 0$$

by concavity of u. Therefore:

$$\frac{\partial s}{\partial R} = -\frac{\beta u'(d)\left[1 - \frac{1}{\sigma(d)}\right]}{u''(c) + \beta R^2 u''(d)}. \tag{3.10}$$

The sign of this derivative depends on whether $\sigma(d) \gtreqless 1$:

$$\frac{\partial s}{\partial R} \begin{cases} > 0 & \text{if } \sigma > 1 \text{ (substitution effect dominates)} \\ = 0 & \text{if } \sigma = 1 \text{ (log-utility: effects cancel)} \\ < 0 & \text{if } \sigma < 1 \text{ (wealth effect dominates)} \end{cases}$$

Economic interpretation:

Two opposing forces determine the sign:

- **Substitution effect**: A higher interest rate makes future consumption cheaper relative to present consumption, encouraging more savings.
- **Wealth effect**: A higher interest rate means that the same savings yield more future consumption, so individuals can save less and still achieve their desired consumption when old.

When $\sigma > 1$, individuals are willing to substitute consumption across time, and the substitution effect dominates. When $\sigma < 1$, individuals prefer smooth consumption paths, and the wealth effect dominates. When $\sigma = 1$ (log-utility), both effects exactly cancel.

3.5 Temporary equilibrium

Before turning to the intertemporal equilibrium and the analysis of the steady state, we study the temporary equilibrium that takes place every period.

Recall that firms use capital and labour in a perfectly competitive environment, paying factors their marginal products as derived in Equation 3.2 and Equation 3.3.

1. **Labour market equilibrium:** *Only* young individuals supply labour, and they do so *inelastically* (one unit each). During period t there are N_t young agents, hence labour supply equals N_t. Equating this to labour demand from firms determines the wage rate:

$$w_t = f(k_t) - f'(k_t)k_t \equiv \omega(k_t).$$

2. **Capital market:** *Only* old individuals own capital (from their savings when young). The capital stock at time t equals the savings of the young generation from period $t-1$:

$$K_t = N_{t-1}s_{t-1}.$$

   Since firms operate competitively with zero profits, they pay the rental rate:

$$R_t = f'(k_t).$$

3. **Goods market:** Total production at time t is:

$$Y_t = F(K_t, N_t) = N_t f(k_t).$$

   Total demand for goods comes from:

   - Old generation consumption: $N_{t-1}d_t$
   - Young generation consumption and savings: $N_t(c_t + s_t)$

Market clearing requires:

$$Y_t = N_{t-1}d_t + N_t(c_t + s_t).$$

Definition: A temporary equilibrium is a set $w_t, R_t, K_t, L_t, Y_t, k_t, c_t, s_t, d_t$ that satisfies:

$$\begin{aligned}
w_t &= \omega(k_t), \\
R_t &= f'(k_t), \\
L_t &= N_t, \\
Y_t &= N_t f(k_t), \\
Y_t &= N_{t-1}d_t + N_t(c_t + s_t), \\
c_t &= w_t - s_t, \\
s_t &= s(\omega(k_t), R_{t+1}), \\
d_t &= R_t s_{t-1}.
\end{aligned}$$

The existence of a temporary equilibrium is guaranteed because all functions are continuous and well-defined.

3.6 Intertemporal equilibrium with perfect foresight

The equilibrium equation that links consecutive periods is the **capital accumulation equation**. In particular, the savings of young individuals at period t are transformed into productive capital at $t + 1$.

3.6.1 From aggregate to per-capita capital accumulation

Note: In the OLG model, it is essential to carefully distinguish between aggregate and per-capita variables. We work through this transformation step by step.

Step 1: Aggregate capital accumulation

The capital stock at time $t+1$ equals the total savings of all young individuals at time t:

$$K_{t+1} = N_t s_t.$$

Since there are N_t young individuals at time t, each saving s_t, total savings equal $N_t s_t$.

Step 2: Definition of per-capita capital

The capital-labour ratio (per-capita capital) at time $t+1$ is defined as:

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}}.$$

Important: Note that we divide by N_{t+1} (the number of young workers at $t+1$), not by N_t. This is because k_{t+1} represents capital per young worker at time $t+1$.

Step 3: Combining both equations

Substituting $K_{t+1} = N_t s_t$ into the definition of k_{t+1}:

$$k_{t+1} = \frac{K_{t+1}}{N_{t+1}} = \frac{N_t s_t}{N_{t+1}}.$$

Step 4: Using population growth

Recall from the preliminaries that $N_{t+1} = N_t(1+n)$, where n is the population growth rate.Substituting:

$$k_{t+1} = \frac{N_t s_t}{N_t(1+n)} = \frac{s_t}{1+n}.$$

Therefore, the per-capita capital accumulation equation is:

$$k_{t+1} = \frac{1}{1+n} s_t. \tag{3.11}$$

Interpretation: This equation shows that per-capita capital growth depends on:

1. Individual savings s_t (the numerator),
2. Population growth n (the denominator).

Even if individuals save a constant amount, per-capita capital can fall if population grows too fast.

3.6.2 The intertemporal equilibrium

Now we incorporate the equilibrium conditions from the markets. Young individuals at time t choose savings based on:

1. Their wage income: $w_t = \omega(k_t)$
2. Their expectation of the future interest rate: $R_{t+1} = f'(k_{t+1})$

Under perfect foresight, individuals correctly anticipate R_{t+1}. Therefore, $s_t = s(\omega(k_t), f'(k_{t+1}))$, and the capital accumulation equation becomes:

$$k_{t+1} = \frac{1}{1+n} s(\omega(k_t), f'(k_{t+1})). \tag{3.12}$$

Definition: Given an initial capital stock $k_0 = K_0/N_{-1}$, an **intertemporal equilibrium with perfect foresight** is a sequence of temporary equilibria
$k_{t_{t=0}}^{\infty}$ that satisfies Equation 3.12 for all $t \geq 0$.

Key observation: Equation Equation 3.12 is an **implicit** equation for k_{t+1} as a function of k_t, because k_{t+1} appears on both sides (inside the savings function through $f'(k_{t+1})$).

3.6.3 Why explicit solutions are difficult

For general utility and production functions, we cannot write an explicit formula $k_{t+1} = g(k_t)$.This is because:

1. The savings function $s(w, R)$ depends on the future interest rate $R_{t+1} = f'(k_{t+1})$,
2. This creates an implicit relationship that typically cannot be solved algebraically.

Exception: Under **log-utility**, savings are independent of the interest rate (Equation 3.8), so $s_t = \frac{\beta}{1+\beta} w_t$ In this case, we can write an explicit capital accumulation equation:

$$k_{t+1} = \frac{1}{1+n} \cdot \frac{\beta}{1+\beta} \omega(k_t). \tag{3.13}$$

This is a major simplification that we will exploit when analyzing dynamics and steady states.

3.6.4 Existence and uniqueness (brief remarks)

The implicit nature of Equation 3.12 raises questions about existence and uniqueness:

- Does a solution k_{t+1} exist for every k_t?
- Is this solution unique?

For our purposes, we note the following results (see Croix and Michel (2009, Ch. 2) for detailed proofs):

1. **Existence:** Under our assumptions (OLG 1-4), at least one temporary equilibrium exists for any $k_t \geq 0$.
2. **Uniqueness:** Uniqueness is guaranteed if the intertemporal elasticity of substitution is sufficiently large. A sufficient condition is $\sigma \geq 1$. When $\sigma < 1$, multiple equilibria can arise, leading to indeterminacy and potential coordination failures.

For the remainder of this chapter, we assume $\sigma \geq 1$, which ensures:

- Unique temporary equilibria,
- Monotonic dynamics: $k_{t+1} = g(k_t)$ is a well-defined function,
- Tractable analysis of steady states and stability.

3.7 Steady states

A **steady state** is a capital-labour ratio $\bar{k}$ that remains constant over time: if $k_t = \bar{k}$, then $k_{t+1} = \bar{k}$. From Equation 3.12, a steady state satisfies:

$$\bar{k} = \frac{1}{1+n} s(\omega(\bar{k}), f'(\bar{k})). \tag{3.14}$$

This equation may have multiple solutions. We consider two important cases.

3.7.1 The autarky steady state: $\bar{k} = 0$

If $f(0) = 0$ (production requires capital), then $\bar{k} = 0$ is always a steady state. When $k_t = 0$:

- Wages are zero: $\omega(0) = 0$,
- Young individuals have no income, hence cannot save: $s_t = 0$,
- Next period's capital remains zero: $k_{t+1} = 0$.

This is called the **autarky steady state** or **poverty trap**. Whether the economy converges to this state depends on the initial conditions and the stability properties.

3.7.2 Interior steady states: $\bar{k} > 0$

The economy may also have positive steady states where $\bar{k} > 0$. Unlike the autarky state, these steady states feature positive production, consumption, and welfare. Finding interior steady states analytically requires solving Equation 3.14, which is typically impossible for general functions. However, with specific functional forms, we can make progress.

Similarly, the Cobb-Douglas case $\rho \to 0$ also has zero as a steady state. However, it is unstable.

3.7.3 Example: Cobb-Douglas production and log-utility

We now work through the complete analysis for the benchmark case: Cobb-Douglas production with log-utility. This combination allows us to derive explicit solutions and fully characterize the dynamics.

Setup:

- Production: $f(k_t) = k_t^\alpha$, $\alpha \in (0, 1)$
- Utility: $u(c) = \log(c)$

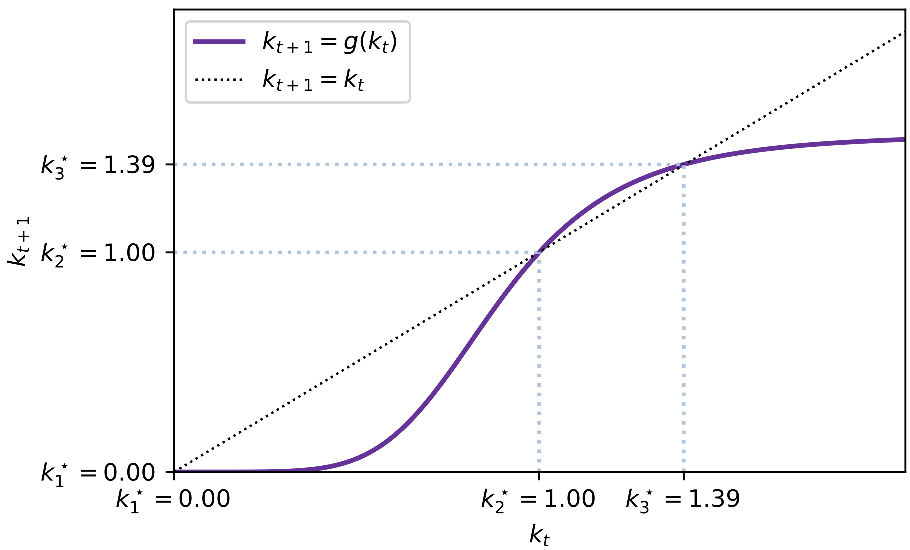


Figure 3.1: Autarky steady state and two positive steady states

- Wages: $w_t = (1-\alpha)k_t^{\alpha}$ (from **?@eq-cobb_douglas_prices**)
- Savings: $s_t = \frac{\beta}{1+\beta} w_t$ (from Equation 3.8)

Capital accumulation:

Combining Equation 3.13 with the Cobb-Douglas wage:

$$k_{t+1} = \frac{1}{1+n} \cdot \frac{\beta}{1+\beta} \cdot (1-\alpha)k_t^{\alpha}.$$

Define $\phi \equiv \frac{\beta(1-\alpha)}{(1+n)(1+\beta)}$, then:

$$k_{t+1} = g(k_t) = \phi k_t^{\alpha}. \tag{3.15}$$

This is an explicit, autonomous difference equation. The dynamics are fully determined by the function $g(k_t)$.

Finding steady states:

A steady state $\bar{k}$ satisfies $\bar{k} = g(\bar{k}) = \phi \bar{k}^{\alpha}$. Rearranging:

$$\bar{k}^{1-\alpha} = \phi.$$

This equation has two solutions:

1. **Autarky:** $\bar{k}_1 = 0$
2. **Interior:** $\bar{k}_2 = \phi^{\frac{1}{1-\alpha}} = \left[\frac{\beta(1-\alpha)}{(1+n)(1+\beta)}\right]^{\frac{1}{1-\alpha}}$

The interior steady state $\bar{k}_2$ is positive and economically meaningful.

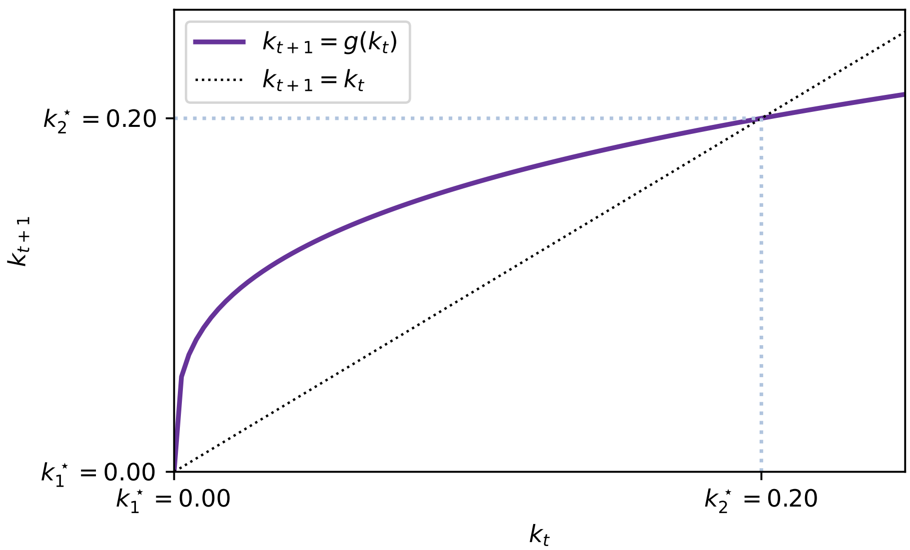


Figure 3.2: Steady states in the Cobb-Douglas with log-utility case

3.8 Stability analysis: The Hartman-Grobman theorem

To determine whether the economy converges to a steady state, we analyze local stability using the **Hartman-Grobman theorem**. This theorem states that the local behavior of a nonlinear dynamical system near a steady state is determined by its **linearization** at that point.

For a one-dimensional discrete-time system $k_{t+1} = g(k_t)$, the linearization around a steady state $\bar{k}$ is:

$$k_{t+1} - \bar{k} \approx g'(\bar{k})(k_t - \bar{k}).$$

The **eigenvalue** of this linear system is simply $\lambda = g'(\bar{k})$. The steady state is:

- **Locally stable** (attractive) if $|\lambda| < 1$,
- **Unstable** (repelling) if $|\lambda| > 1$,
- **Indeterminate** if $|\lambda| = 1$ (requires higher-order analysis).

Note: For our one-dimensional system, the "Jacobian matrix" is just the scalar $g'(\bar{k})$, and the "eigenvalue" is this scalar itself.

3.8.1 Stability of the autarky steady state

For $\bar{k}_1 = 0$:

$$g'(k) = \phi \alpha k^{\alpha-1}.$$

Taking the limit as $k \to 0$:

$$\lim_{k \to 0} g'(k) = \lim_{k \to 0} \phi \alpha k^{\alpha-1} = +\infty$$

because $\alpha - 1 < 0$. Since $|g'(0)| = +\infty > 1$, the autarky steady state is **locally unstable**.

Interpretation: If the economy starts with any $k_0 > 0$ (even arbitrarily small), it will move away from $k = 0$ toward the interior steady state.

3.8.2 Stability of the interior steady state

For $\bar{k}_2 = \phi^{\frac{1}{1-\alpha}}$:

$$g'(\bar{k}_2) = \phi\alpha\bar{k}_2^{\alpha-1}.$$

Substitute $\bar{k}_2^{1-\alpha} = \phi$, which implies

$$\bar{k}_2^{\alpha-1} = \left(\bar{k}_2^{1-\alpha}\right)^{-1} \cdot \bar{k}_2^{0} = \frac{1}{\phi}.$$

Therefore:

$$g'(\bar{k}_2) = \phi\alpha \cdot \frac{1}{\phi} = \alpha.$$

Since $\alpha \in (0,1)$, we have $|g'(\bar{k}_2)| = \alpha < 1$. The interior steady state is **locally stable**.

Interpretation: For any initial capital k_0 in a neighborhood of $\bar{k}_2$, the economy converges to $\bar{k}_2$ over time.

3.8.3 Global dynamics

We can say more about global behavior:

1. **For any $k_0 > 0$:** The function $g(k) = \phi k^\alpha$ is increasing and concave. Since the autarky steady state is unstable and the interior steady state is stable, any initial $k_0 > 0$ converges to $\bar{k}_2$.

2. **Monotonic convergence:**

   - If $k_0 < \bar{k}_2$: The sequence
     k_t is increasing and converges from below.
   - If $k_0 > \bar{k}_2$: The sequence
     k_t is decreasing and converges from above.

3. **Rate of convergence:** The speed of convergence is determined by α. Smaller α (lower capital share) implies faster convergence.

Graphical representation: The dynamics are illustrated by plotting $k_{t+1} = \phi k_t^\alpha$ against the 45-degree line $k_{t+1} = k_t$.

Single, non-autarky steady state This steady state is characterized by a unique positive capital level $\bar{k} > 0$. This configuration requires $f(0) > 0$. With a CES production function, capital and labour must be substitutes, this is, $< 0\rho < 1$: $(\alpha k^\rho + (1-\alpha))^{\frac{1}{\rho}}$.

Only an autarky steady state In this case, $f(0) = 0$ is required so that the autarky steady state can exist. Moreover, it requires $f(k_{t+1}) < f(k_t)$. This type of setup requires $\rho < 0$, this is, complementarity between capital and labor and low levels of output given the inputs.

3.9 Application: Fertility and education choices

So far, we have treated the population growth rate n as exogenous. However, households make deliberate choices about fertility and investment

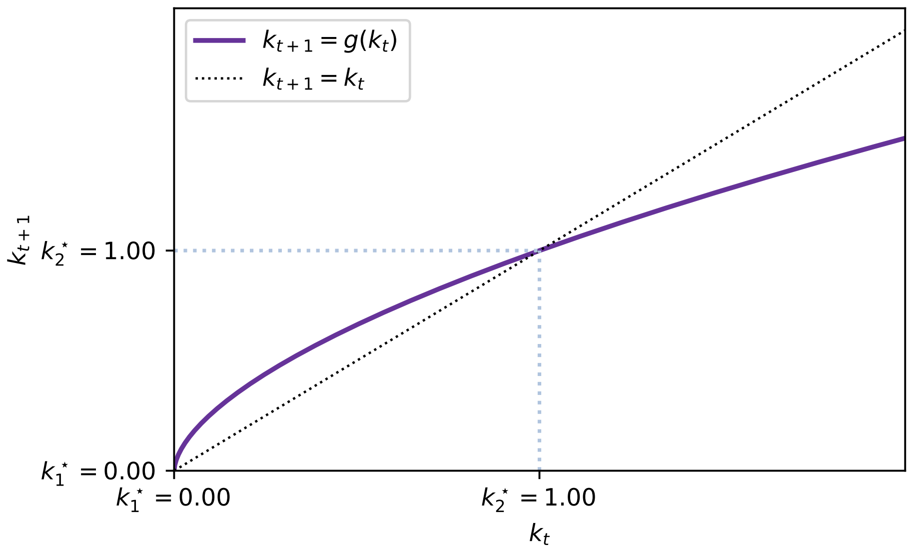


Figure 3.3: Autarky steady state and a positive steady state

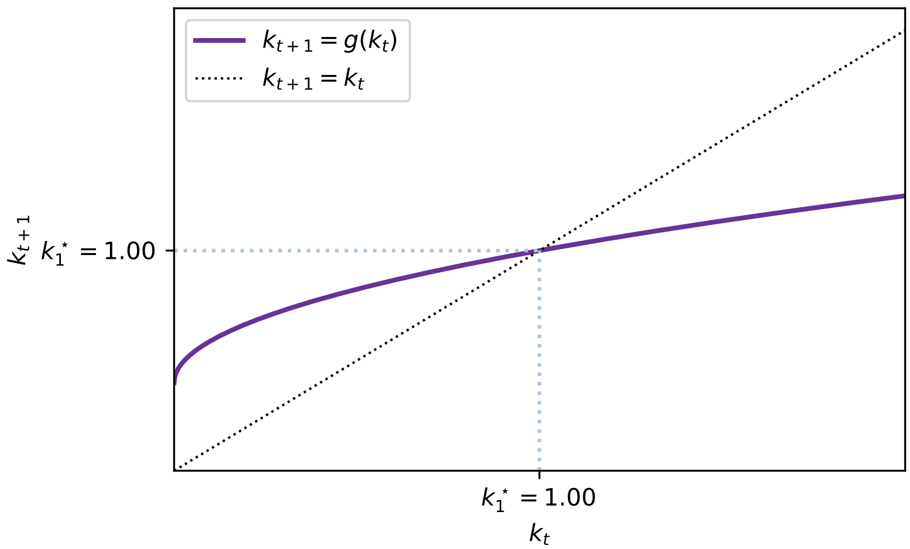


Figure 3.4: Single, non-autarky steady state

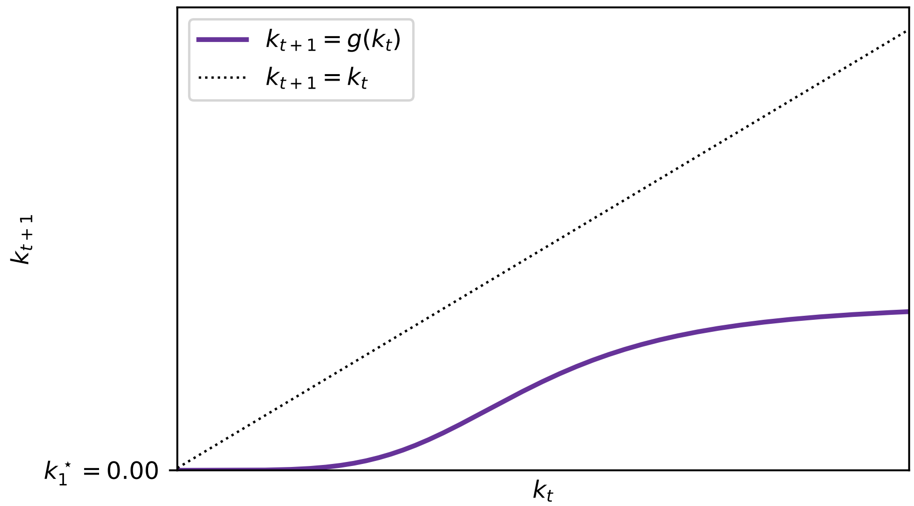


Figure 3.5: Only an autarky steady state

in children's human capital. These decisions are central to understanding economic development and the demographic transition.[2]

We now extend the model to incorporate the **quality-quantity trade-off**: parents must decide between having more children (quantity) and investing more in each child's education (quality). In this version, we abstract from capital accumulation to focus on human capital dynamics.

3.9.1 Model setup

Production: Output is produced using human capital and raw labor:

$$Y_t = H_t^{\alpha} L_t^{1-\alpha} \tag{3.16}$$

[2]For a comprehensive review of the economics of fertility and family economics, see Doepke et al. (2023).

where H_t is the aggregate level of human capital in the economy and L_t is raw labor input. The output per worker is:

$$y_t = y_t^{\alpha}.$$

Since workers own both factors of production (their human capital and their raw labor), they receive the entire output as wages:

$$w_t = h_t^{\alpha}. \tag{3.17}$$

Alternatively, if we let L_t be the raw labor (hours) supplied by workers and each supplies l_t units, then $L_t = N_t l_t$ where N_t is the number of workers. Similarly, if each worker has human capital h_t, then $H_t = h_t L_t = h_t N_t l_t$. Thus, total production is:

$$Y_t = H_t^{\alpha} L_t^{1-\alpha} = (h_t N_t l_t)^{\alpha} (N_t l_t)^{1-\alpha} = h_t^{\alpha} (N_t l_t) = h_t^{\alpha} L_t.$$

Thus, output per worker is $y_t = h_t^{\alpha}$. Because workers own their human capital, they receive the entire output as wages: $w_t = h_t^{\alpha}$.

> **Note**
>
> If we wanted to have two factors of production (human capital and raw labor) paid separately, under a Cobb-Douglas production function, like $Y = L^{\beta} H^{1-\beta}$, then the wage paid to raw labor would be:
>
> $$w_{raw} = \beta \left(\frac{Y}{L} \right).$$
>
> and the return to human capital would be:
>
> $$r_{human} = (1 - \beta) \left(\frac{Y}{H} \right).$$
>
> Because a worker supplies 1 unit of raw labor and has h units of human capital, their total income would be:

$$\begin{aligned} w =& w_{raw} + r_{human} h = \beta \left(\frac{Y}{L} \right) + (1-\beta) \left(\frac{Y}{H} \right) h \\ =& \beta \left(\frac{Y}{L} \right) + (1-\beta) \left(\frac{Y}{hL} \right) h = y. \end{aligned}$$

Preferences: Households live for one period and care about their own consumption, the number of children, and the human capital of their children:

$$U_t = \log(c_t) + \gamma \log(n_t e_{t+1}^{\beta}) \tag{3.18}$$

where:

- c_t is consumption,
- $n_t \geq 0$ is the number of children,
- $e_{t+1} \geq 0$ is the education (human capital) investment per child,
- $\gamma > 0$ measures the weight on children,
- $\beta \in (0, 1)$ captures the relative importance of child quality.

Budget constraint: Raising children requires time. Each child requires $\phi > 0$ units of parental time, reducing time available for work. The household's time endowment is normalized to 1, so time available for work is $1 - \phi n_t$.

Parents allocate their wage income between consumption and education investment:

$$w_t(1 - \phi n_t) = c_t + n_t e_{t+1}. \tag{3.19}$$

Feasibility: We require $\phi n_t \leq 1$ to ensure non-negative labor supply.

3.9.2 Household optimization

Substituting the budget constraint into the utility function:

$$U_t = \log[w_t(1-\phi n_t) - n_t e_{t+1}] + \gamma \log(n_t e_{t+1}^{\beta}).$$

Using properties of logarithms:

$$U_t = \log[w_t(1-\phi n_t) - n_t e_{t+1}] + \gamma \log(n_t) + \gamma\beta \log(e_{t+1}). \tag{3.20}$$

First-order conditions:

1. **With respect to n_t:**

$$\frac{w_t\phi + e_{t+1}}{w_t(1-\phi n_t) - n_t e_{t+1}} = \frac{\gamma}{n_t} \tag{3.21}$$

2. **With respect to e_{t+1}:**

$$\frac{n_t}{w_t(1-\phi n_t) - n_t e_{t+1}} = \frac{\gamma\beta}{e_{t+1}} \tag{3.22}$$

Let $c_t = w_t(1-\phi n_t) - n_t e_{t+1}$. Then the FOCs become:

From Equation 3.21:

$$\frac{w_t\phi + e_{t+1}}{c_t} = \frac{\gamma}{n_t} \implies \gamma c_t = n_t(w_t\phi + e_{t+1}). \tag{3.23}$$

From Equation 3.22:

$$\frac{n_t}{c_t} = \frac{\gamma\beta}{e_{t+1}} \implies \gamma\beta c_t = n_t e_{t+1}. \tag{3.24}$$

3.9.3 Solving for fertility and education

From Equation 3.23 and Equation 3.24, we can isolate the denominator c_t in both.

From Equation 3.23:

$$c_t = \frac{n_t(w_t\phi + e_{t+1})}{\gamma}$$

And from Equation 3.24:

$$c_t = \frac{n_t e_{t+1}}{\gamma\beta}$$

Thus, equating these two expressions for c_t:

$$\frac{n_t(w_t\phi + e_{t+1})}{\gamma} = \frac{n_t e_{t+1}}{\gamma\beta}.$$

Dividing both sides by n_t (assuming $n_t > 0$, which will always be the case due to the logarithm in the utility) and multiplying through by γ:

$$w_t\phi + e_{t+1} = \frac{e_{t+1}}{\beta}.$$

Rearranging:

$$e_{t+1} = \frac{\beta\phi}{1-\beta} w_t.$$

Next, substituting e_{t+1} back into either expression for c_t (we use Equation 3.24):

$$\frac{w_t\phi + e_{t+1}}{w_t(1-\phi n_t) - n_t e_{t+1}} = \frac{\gamma}{n_t}.$$

Thus,

$$\frac{w_t\phi + \frac{\beta w_t \phi}{1-\beta}}{w_t(1-\phi n_t) - n_t\frac{\beta w_t\phi}{1-\beta}} = \frac{\gamma}{n_t}.$$

Dividing numerator and denominator on the left-hand side by w_t:

$$\frac{\phi + \frac{\beta\phi}{1-\beta}}{(1-\phi n_t) - n_t\frac{\beta\phi}{1-\beta}} = \frac{\gamma}{n_t}.$$

Multiplying the left-hand side numerator and denominator by $(1-\beta)$ to clear the fraction:

$$\frac{\phi(1-\beta)+\beta\phi}{(1-\beta)(1-\phi n_t) - n_t\beta\phi} = \frac{\gamma}{n_t}.$$

Rearranging:

$$\phi n = \gamma(1-\beta)(1-\phi n) - \gamma n\beta\phi.$$

Collecting terms involving n_t on the left-hand side and isolating n_t:

$$n_t = \frac{\gamma(1-\beta)}{(1+\gamma)\phi}.$$

Key results:

1. **Fertility is constant:** $n_t = \frac{\gamma(1-\beta)}{(1+\gamma)\phi}$ depends only on parameters.
2. **Education is proportional to wages:** $e_{t+1} = \frac{\beta\phi}{1-\beta}w_t$. As the economy develops and wages rise, parents invest more in each child's human capital.

3.9.4 Human capital dynamics

Since wages equal $w_t = h_t^\alpha$ and human capital equals the education received in the previous period ($h_t = e_t$), we have:

$$e_{t+1} = \frac{\beta\phi}{1-\beta} h_t^\alpha. \tag{3.25}$$

This is a first-order difference equation in education/human capital. In the next period:

$$h_{t+1} = e_{t+1} = \frac{\beta\phi}{1-\beta} h_t^\alpha. \tag{3.26}$$

Let $A = \frac{\beta\phi}{1-\beta}$. Then:

$$h_{t+1} = A h_t^\alpha. \tag{3.27}$$

Steady state: At steady state, $h_{t+1} = h_t = h^*$:

$$h^* = A(h^*)^\alpha \implies (h^*)^{1-\alpha} = A \implies h^* = A^{\frac{1}{1-\alpha}}. \tag{3.28}$$

Stability: Deriving the dynamic equation Equation 3.27 and evaluating at the steady state:

$$\frac{\partial h_{t+1}}{\partial h_t} = A\alpha h_t^{\alpha-1}.$$

At steady state:

$$\left.\frac{\partial h_{t+1}}{\partial h_t}\right|_{h_t=h^*} = A\alpha(h^*)^{\alpha-1} = A\alpha \cdot \frac{1}{A} = \alpha.$$

Since $\alpha \in (0,1)$, this converges to the steady state from any initial condition.

3.9.5 GDP per worker dynamics

Since $y_t = h_t^\alpha$, GDP per worker evolves according to:

$$y_t = h_t^\alpha = \left(Ah_{t-1}^\alpha\right)^\alpha = A^\alpha h_{t-1}^{\alpha^2}. \tag{3.29}$$

More generally, from $h_t = Ah_{t-1}^\alpha$:

$$y_t = h_t^\alpha.$$

At steady state:

$$y^\star = (h^\star)^\alpha = A^{\frac{\alpha}{1-\alpha}}. \tag{3.30}$$

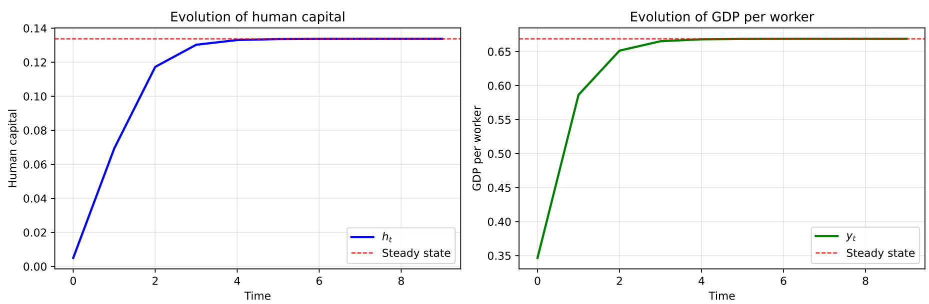


Figure 3.6: Human capital and GDP dynamics

3.9.6 Discussion

This application illustrates how the OLG framework can be extended to analyze human capital accumulation through education choices:

1. **Endogenous growth:** Unlike models with exogenous human capital, here education investments respond to wages, creating a feedback loop between human capital and productivity.

2. **Quality-quantity trade-off:** Parents optimally choose fertility $n_t = \frac{\gamma(1-\beta)}{(1+\gamma)\phi}$ and education per child $e_{t+1} = \frac{\beta\phi w_t}{1-\beta}$. The parameter β governs the relative importance of quality vs. quantity.

3. **Convergence:** The economy converges to a steady state level of human capital $h^* = \left(\frac{\beta\phi}{1-\beta}\right)^{\frac{1}{1-\alpha}}$ and GDP per worker $y^* = (h^*)^\alpha$.

4. **Policy implications:** Policies that reduce the time cost of children (ϕ) or increase the value of education (β) can raise long-run human capital and income levels.

For further reading on these topics and empirical evidence, see the comprehensive survey by Doepke et al. (2023).

4 de la Croix and Dottori (2008)

This section discusses Croix and Dottori (2008), using a simplified model to illustrate the key points.

4.1 Introduction

This paper analyses the very distinct population patterns of two remote islands in the Pacific Ocean: Eastern Island and Tikopia. While Tikopians managed to control population growth and natural resources usage, the inhabitants of Eastern Island engaged in clan competition for the control of resources, leading to overpopulation and overexplotation of resources.

A key aspect of this paper is the especial role of fertility. Most papers assume that parents derive some utility from having children. However, in de la Croix and Dottori, fertility is the result of a Nash bargaining process for the control of resources. In particular, a larger population, facilitated by having more children, raises the value of the fallback option during the negotiation process. Consequently, individuals optimally decide to have more children, because this implies a better bargaining position. In this sense, equilibrium-level fertility rates result from the complementarities between different groups' fertility decisions. However, the externalities of a higher fertiltiy rate are not internalised and, in the long run, population explodes, leading to a natural catastrophe.

4.1.1 Historical data

Based on data from archaeological studies, it has been estimated that the population of Eastern Island increased very little between 400CE (100 people by that time) and 110CE, and from then on, it exploded, reaching 10000 people during 1400-1600CE. The effects of the population race could be perceived by 1600CE: food consumption declined and population plummeted during the 17th century. By 1772, when Europeans arrived at Eastern Island, the total population was around 3000 people. In parallel, data about forests in Eastern Island indicate that upon the arrival of the first settlers during 400CE, tree-cutting begun. By 1400CE, deforestation had reached its peak and when the Europeans arrived, there were basically no trees in the island.

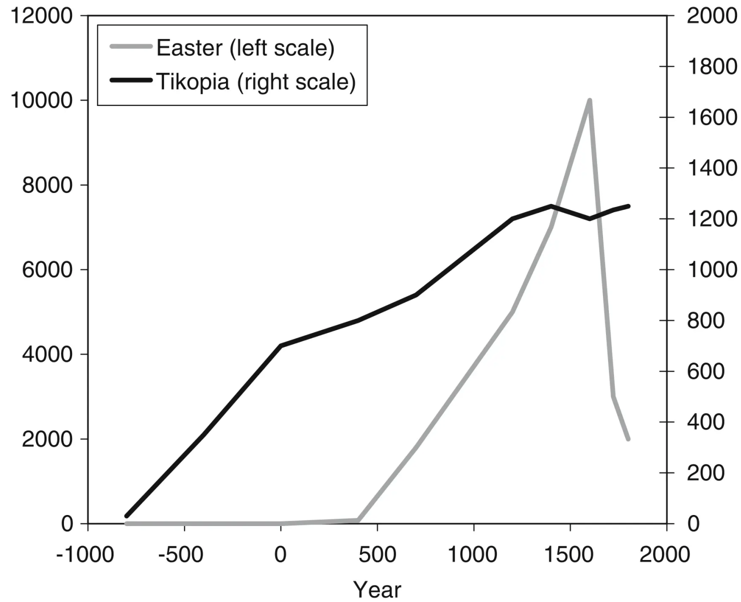


Fig. 1 Population of Easter Island and Tikopia

Figure 4.1: Figure 1 in de la Croix and Dottori, 2008

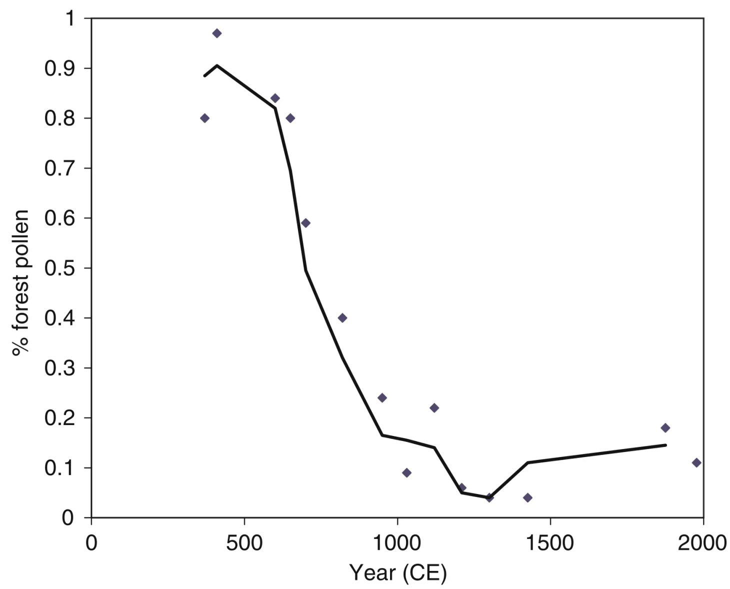


Fig. 2 Forest coverage on Easter Island

Figure 4.2: Figure 2 in de la Croix and Dottori, 2008

Meanwhile, Tikopia was settled around 900BCE and people lived by slash and burn agriculture. By 100BCE, due to decreasing returns from natural resources, pig breeding began, lasting until the 17th century, when Tikopians abandoned it because pigs required too many resources. Total population stabilized at around 1200 people and was kept at that level by purposeful mechanisms: celibacy, abortion, infanticide, sea exploration by young males, etc.

4.2 The model

De la Croix and Dottori work using the OLG framework. In the paper, agents live for two periods. However, important decisions are taken at the clan level, which acts as a representative agent. Clans (and individuals) are rational, have perfect foresight and take the actions of the other clans as given. The timing is as follows:

1. Each clan chooses its fertility level,
2. A Nash-Cournot fertility equilibrium level arises,
3. Crops are cultivated and shared between clans following a non-cooperative bargaining process.

For simplicity, the island is populated by two opposed clans, all individuals belong to one clan only and they cannot change clan.

4.2.1 Preferences

Clan i at time t consists of $N_{i,t}$ adults. Adults work, support their parents and have children. Old agents only consume what their children provide for them.[1] Total utility is given by:

$$U_{i,t} = c_{i,t} + \beta d_{i,t+1},$$

where $c_{i,t}, d_{i,t+1}$ represents consumption when adult and old, respectively.

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

4.2.2 Budget

The income of an adult agent is $y_{i,t}$. Each adult has to support his parents by giving them some resources. However, support for parents is not linear, but rather it depends on the number of siblings. In particular, each sibling contributes the following share of his income:

$$\frac{\tau}{1+n_{i,t-1}},$$

where $\tau \in (0,1)$. Clearly, the contribution decreases with the number of siblings.

Consequently, an agent in his old age who had $n_{i,t}$ children receives, as **total** old age support,

$$d_{i,t+1} = n_{i,t}\frac{\tau}{1+n_{i,t}}y_{i,t+1}.$$

Lastly, since total income is distributed between consumption and supporting parents $\left(y_{i,t} = c_{i,t} + \frac{\tau}{1+n_{i,t-1}}y_{i,t}\right)$, consumption when young is just:

$$c_{i,t} = \left(1 - \frac{\tau}{1+n_{i,t-1}}\right)y_{i,t}.$$

4.2.3 Population

The population of each clan evolves according to the chosen fertility level:

$$N_{i,t+1} = N_{i,t}n_{i,t}.$$

4.2.4 Production

Production depends only on land. The amount of land is fixed at L, and total factor productivity depends on the available natural resources R_t.

$$Y_{i,t} = A(R_t)L.$$

The dynamics of resources follow the paper by Matsumoto (2002):

$$R_{t+1} = \left(1 + \delta - \delta \frac{R_t}{K} - b(N_{1,t} + N_{2,t})\right) R_t,$$

where $K > 0$ is the carrying capacity (maximum possible number of resources), $\delta > 0$ is the growth rate of resources while $b > 0$ measures the effect of population on resources.

4.2.4.1 Crop-sharing

We denote by θ_t the share of crops Y_t that Group 1 appropriates. Therefore, each adult in Groups 1 and 2 obtains:

$$y_{1,t} = \theta_t \frac{Y_t}{N_{1,t}},$$

$$y_{2,t} = (1 - \theta_t) \frac{Y_t}{N_{2,t}}.$$

There are no property rights on the island, and groups have to bargain to decide how to split the total porduction Y_t. This bargaining process is non-cooperative and, if no agreement is reached, clans will battle to appropriate the entire production.

4.3 Bargaining

Bargaining takes place under Nash-bargaining, and the outcome of the process solves:

$$(U_{1,t} - \bar{U}_{1,t})^{\gamma}(U_{2,t} - \bar{U}_{2,t})^{1-\gamma},$$

where $U_{1,t}$ is what Group 1 shall receive and $\bar{U}_{1,t}$ is the fall-back option of Group 1, this is, what Group 1 receives is there is no agreement. When there is no agreement between Group 1 and Group 2, the clans fight and the winner takes all. The probability that Group 1 wins the war, denoted by π_t, depends on its size.

$$\pi_t = \frac{N_{1,t}}{N_{1,t} + N_{2,t}}.$$

So, the more adults in one group, the more likely it is to win the war. From the equation, it is clear that clans have an incentive to increase their population: it helps win the war (if it happens) and provides them with a better bargaining position by raising $\bar{U}_{i,t}$.

Suppose than clans reach an agreement θ_t on how to share crops: θ_t goes to Group 1, and the remining $1 - \theta_t$ goes to Group 2. Then, the indirect utility of an individual is given by:

$$U_{1,t} = \left(1 - \frac{\tau}{1 + n_{1,t-1}}\right)\frac{\theta_t Y_t}{N_{1,t}} + \beta\frac{n_{1,t}\tau}{1 + n_{1,t}}\frac{\theta_{t+1}Y_{t+1}}{N_{1,t+1}},$$

$$U_{2,t} = \left(1 - \frac{\tau}{1 + n_{2,t-1}}\right)\frac{(1-\theta_t)Y_t}{N_{2,t}} + \beta\frac{n_{2,t}\tau}{1 + n_{2,t}}\frac{(1-\theta_{t+1})Y_{t+1}}{N_{2,t+1}}.$$

If, instead, there is no agreement, clans fight, and the winner takes the entire production. Since π_t denotes the probability that Group 1 wins the fight, the fall-back utilities are given by:

$$\bar{U}_{1,t} = \pi_t \left(1 - \frac{\tau}{1+n_{1,t-1}}\right) \frac{Y_t}{N_{1,t}} + \beta \frac{n_{1,t}\tau}{1+n_{1,t}} \frac{\theta_{t+1} Y_{t+1}}{N_{1,t+1}},$$

$$\bar{U}_{2,t} = (1-\pi_t) \left(1 - \frac{\tau}{1+n_{2,t-1}}\right) \frac{Y_t}{N_{2,t}} + \beta \frac{n_{2,t}\tau}{1+n_{2,t}} \frac{(1-\theta_{t+1}) Y_{t+1}}{N_{2,t+1}}.$$

The difference between $U_{i,t}$ and $\bar{U}_{i,t}$ is:

$$U_{1,t} - \bar{U}_{1,t} = \left(1 - \frac{\tau}{1+n_{1,t-1}}\right) (\theta_t - \pi_t) \frac{Y_t}{N_{1,t}},$$

$$U_{2,t} - \bar{U}_{2,t} = \left(1 - \frac{\tau}{1+n_{2,t-1}}\right) (1 - \theta_t - (1-\pi_t)) \frac{Y_t}{N_{2,t}}.$$

Since, at the time of bargaining, $\left(1 - \frac{\tau}{1+n_{i,t}}\right) \frac{Y_t}{N_{i,t}}$ has already been determined, we can abstract from it in the maximisation.

After substituting, θ_t, is the optimal sharing rule which solves

$$\theta_t = \arg\max \left(\theta_t - \pi_t\right)^{\gamma} \left(1 - \theta_t - (1-\pi_t)\right)^{1-\gamma}.$$

The optimal level θ_t is then:

$$\theta_t = \frac{N_{1,t}}{N_{1,t} + N_{2,t}}.$$

4.4 Fertility

Finally, we can compute the optimal fertility levels for each group by maximising utility. So, Group 1 and Group 2 maximise:

$$\max_{n_{1,t}} \overbrace{\left(1 - \frac{\tau}{1+n_{1,t-1}}\right)\frac{\theta_t Y_t}{N_{1,t}}}^{constant\,at\,t} +$$

$$+\frac{\beta\tau n_{1,t}}{1+n_{1,t}}\left[\frac{\overbrace{N_{1,t}n_{1,t}}^{N_{1,t+1}}}{\underbrace{N_{1,t}n_{1,t}}_{N_{1,t+1}} + \underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}}}\right]\frac{A(R_{t+1})}{\underbrace{N_{t,1}n_{t,1}}_{N_{1,t+1}}}.$$

$$\max_{n_{2,t}} \overbrace{\left(1 - \frac{\tau}{1+n_{2,t-1}}\right)\frac{\theta_t Y_t}{N_{2,t}}}^{constant\,at\,t} +$$

$$+\frac{\beta\tau n_{2,t}}{1+n_{2,t}}\left[1 - \frac{\overbrace{N_{2,t}n_{2,t}}^{N_{2,t+1}}}{\underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}} + \underbrace{N_{2,t}n_{2,t}}_{N_{2,t+1}}}\right]\frac{A(R_{t+1})}{\underbrace{N_{t,2}n_{t,2}}_{N_{2,t+1}}}.$$

The optimum levels of fertility satisfy:

$$n_{1,t}^{\star} = \left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}},$$

$$n_{2,t}^{\star} = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

So, the best course of action for Group 1 is to increase its fertility as Group 2 becomes more populous, and a race for population occurs. Of course, this has implications for the environment, because larger populations are destructive:

$$R_{t+1} = \left(1 + \delta - \delta\frac{R_t}{K} - b(N_{1,t} + N_{2,t})\right) R_t,$$

4.5 Steady state level of population and natural resources

Lastly, we can compute the steady state level of population using the two dynamical equations:

$$n^{\star}_{1,t} = \left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}} \implies N_{1,t+1} = N_{1,t}\left(\frac{N_{2,t}}{N_{1,t}}\right)^{\frac{1}{3}}$$

$$n^{\star}_{2,t} = \left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}} \implies N_{2,t+1} = N_{2,t}\left(\frac{N_{1,t}}{N_{2,t}}\right)^{\frac{1}{3}},$$

It is simpler to solve the corresponding linearised system, which we can obtain by taking logarithms:

$$\tilde{N}_{i,t+1} = \frac{2}{3}\tilde{N}_{i,t} + \frac{1}{3}\tilde{N}_{j,t},$$

where $\tilde{N} = \log N$.

We can obtain the dynamics of the logarithmic system:

$$\tilde{N}_{i,t} = \frac{\tilde{N}_{i,0} + \tilde{N}_{j,0}}{2} + \frac{1}{2}3^{-t}\left(\tilde{N}_{i,0} - \tilde{N}_{j,0}\right).$$

The steady state level of population, for each group is

$$\bar{N}_i = \bar{N}_j = \sqrt{N_{1,0}}\sqrt{N_{2,0}},$$

and the corresponding level of natural resources at the steady state is

$$\bar{R} = K\left(1 - \frac{b(\bar{N}_i + \bar{N}_j)}{\delta}\right).$$

4.5.1 Simulated trajectory

Lastly, we can compute the trajectory of the system for a set of parameters to visualize the evolution of the main variables. For instance, if we take the following parametrisation $N_{1,0} = 9, N_{2,0} = 20, \delta = 0.08, K = 400, b = 0.0012, R_0 = 300$

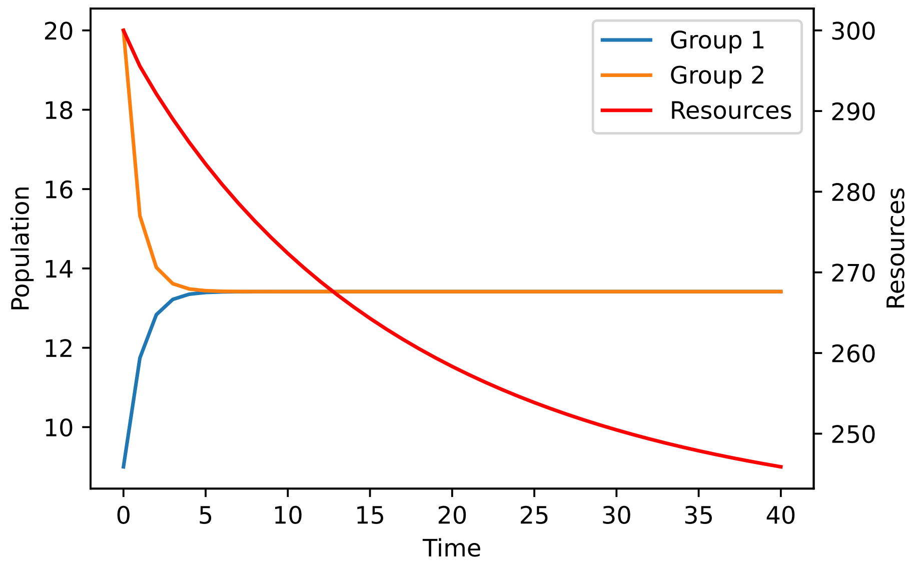


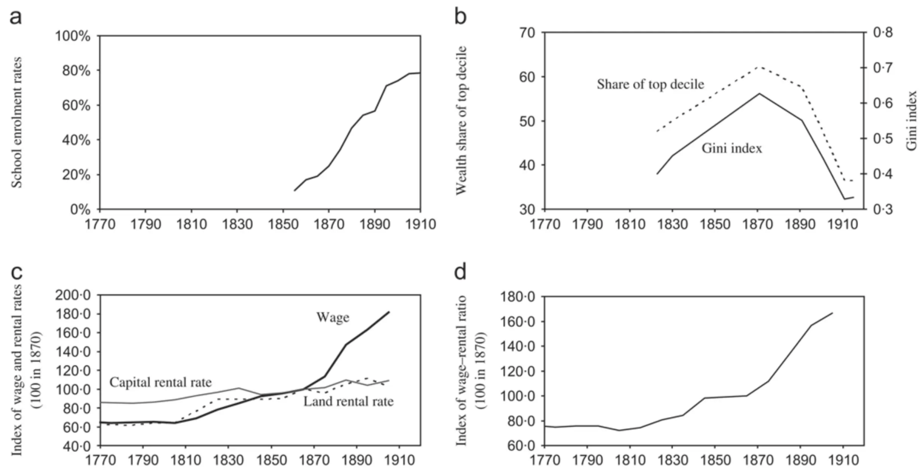
Figure 4.3: Simulated trajectory of populations and resources over time

5 Galor and Moav (2006)

This section discusses the work of Galor and Moav (2006), which presents an OLG model to explain the rise of publicly financed education and the transition from a class structure characterised by capitalists and workers to another one where all individuals are capitalists.

The papers proposes that the demise of the class structure was a deliberate action from the part of the capitalists to sustain their profits as human capital became more and more important in the production process. This is, at some point in time, human capital becomes *really* necessary to produce, and the capitalists find it optimal to tax themselves to finance the education of workers. Doing so, raises the human capital level and allows them to keep their profits.

The paper is set-up after the industrial revolution, let's say around 1850 and is meant to describe the transition of the modern economies between that time and the beginning of the 20th century. The authors motivate the paper by showing that during this time, school enrolment rates increased while, at the same time, inequality decreased: the demise of classes. The theory the authors propose parallels this evolution. Furthermore, they supplement the model with econometrics that are compatible with the predictions of the model.



Source: Flora *et al.* (1983), real wage (Clark, 2005), rental rates (Clark, 2002), real wage (Clark, 2005), and return to capital (Clark, 2002).

FIGURE 1

Schooling, factor prices, and inequality, England 1770–1920. The evolution of (a) the fraction of children aged 5–14 in public primary schools: England, 1855–1920, (b) earnings inequality: England, 1820–1913, (c) wages and rental rates: England, 1770–1920, and (d) the wage–rental ratio: England, 1770–1920

Figure 5.1: Figure 1 in Galor and Moav (2006)

5.1 The model

We present a simplified version of the model using precise functional forms for the human capital accumulation process and the production function. This paper comprises two separate sections: * The general model * The application of the model to a society with two classes. We will follow this approach.

For the moment, the economy comprises *only* one type of individuals. Individuals live for two periods of time: young and adult. Young individuals do not produce and use their time to acquire human capital. If education is provided, human capital accumulation is faster.

5.1.1 Production and prices

A single homogenous good is produced using physical and human capital according to a Cobb-Douglas production function. In particular:

$$Y_t = F(H_t, K_t) = AK_t^{\alpha} H_t^{1-\alpha} = Ak_t^{\alpha}, \quad k_t \equiv \frac{K_t}{H_t}.$$

Given the wage rate per efficiency unit of labour w_t and the return to capital r_t, producers maximise profits by choosing the level of capital K_t and efficiency units of labour H_t, this is,
$K_t, H_t \ = \ \arg\max \left[AH_t k_t^{\alpha} - w_t H_t - r_t K_t\right].$ Considering perfect competition, the inverse demand for each factor is:

$$\begin{aligned} r_t &= f'(k_t) = \alpha A k_t^{\alpha-1} = r(k_t), \\ w_t &= f(k_t) - f'(k_t)k_t = (1-\alpha)Ak_t^{\alpha} = w(k_t). \end{aligned}$$

5.1.2 Individuals and preferences

Every period, a new generation of size 1 is born. Individuals have only parent and each has only one son. Individuals live for two periods: during their youth, they accumulate human capital; and education improves human capital accumulation. Young individuals may receive a positive bequest from their parents on which they earn interests (the bequest is physical capital that is lent to producers). When adults, they supply their human capital as efficiency units of labour; receive interests on their assets and allocate the total income between consumption and a bequest.

The bequest b_t is transferred from parents to children, and the government collects a tax $\tau_t \geq 0$ on it. The remaining $1 - \tau_t$ is saved for future consumption. Physical capital **fully depreciates** between periods.

As mentioned, individuals accumulate human capital during their youth, and if they are provided with education (e_t), human capital accumulation is enhanced. However, even if no education is provided, all young individuals manage to obtain a minimum level of human capital: we set it equal to one. We model human capital accumulation as follows:[1]

$$h_{t+1} = 1 + \frac{e_t}{1 + e_t} = h(e_t).$$

This type of function ensures that under some conditions, investment in education is not optimal; while guaranteeing a minimum level of human capital.

When adults, individuals receive wages on their human capital, as well as the rental price on the bequest (the part not taxed by the government). Therefore, an individual with a bequest b_t and education level e_t has income equal to:

$$I^i_{t+1} = w_{t+1}h(e_t) + (1 - \tau)b^i_t R_{t+1},$$

and, since capital fully depreciates, $R_{t+1} = r_{t+1} = r(k_{t+1})$.

Lastly, the preferences of adults include consumption and a taste for giving bequests. Bequests in the model are a type of luxury good: only when income is large enough, $b_t > 0$. In particular:

$$u^i_t = (1 - \beta) \log(c^i_{t+1}) + \beta \log(\bar{\theta} + b^i_{t+1}),$$

where $\bar{\theta} > 0$ and $\beta \in (0, 1)$. The budget constraint is simple:

$$c^i_{t+1} + b^i_{t+1} = I^i_{t+1}.$$

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

5.1.3 Optimisation

We can easily compute the value of bequests, as a function of the income level. Replacing c_{t+1} in the objective function and taking the derivative with respect to b_{t+1} yields:

$$\frac{\partial}{\partial b^i_{t+1}} = 0 \implies \frac{1-\beta}{b^i_{t+1} - I^i_{t+1}} + \frac{\beta}{b^i_{t+1} + \bar{\theta}} = 0 \implies$$

$$b^i_{t+1} = \begin{cases} \beta(I^i_{t+1} - \theta) & \text{if} \quad I^i_{t+1} > \theta \\ 0 & \text{if} \quad I^i_{t+1} \leq \theta \end{cases}$$

where $\theta \equiv \bar{\theta}\frac{1-\beta}{\beta}$. Hence, when income is relative low, individuals do not give bequests.

5.1.4 Evolution of physical and human capital

Remember that bequests left during period t are the capital of period $t+1$. If B_t is the total amount of bequests left during t, then

$$K_{t+1} = (1-\tau)B_t.$$

The remaining τB_t goes to the government, which uses it to fund education. Population is normalised to 1, therefore, each individual receives education equal to: $e_t = \tau B_t$, and human capital evolves as:

$$H_{t+1} = h(e_t) = h(\tau_t B_t) = 1 + \frac{\tau_t B_t}{1 + \tau_t B_t}.$$

Last, the level of $k_t = \frac{K_t}{H_t}$ is:

$$k_{t+1} = \frac{K_{t+1}}{H_{t+1}} = \frac{(1-\tau_t)B_t}{h(\tau_t B_t)} = \frac{(1-\tau_t)B_t}{1 + \frac{\tau_t B_t}{1+\tau_t B_t}} = k(\tau_t, B_t).$$

5.2 Optimal level of taxation

The paper assumes that the government selects the tax rate that maximises individuals' utility. One important feature of the model is that utility is increasing in income I^i_{t+1}. We can easily check this by rewriting the indirect utility:

$$
\begin{aligned}
u^i_t &= (1-\beta)\log(c^i_{t+1}) + \beta\log(\bar{\theta} + b^i_{t+1}) = \\
&= \begin{cases} (1-\beta)\log(I^i_{t+1} - \beta I^i_{t+1} + \beta\theta) + \beta\log(\beta I^i_{t+1} - \beta\theta) & \text{if} \quad I^i_{t+1} > \theta \\ (1-\beta)\log(I^i_{t+1}) + \beta\log(\bar{\theta}) & \text{if} \quad I^i_{t+1} \leq \theta \end{cases}
\end{aligned}
$$

which is increasing in I^i_{t+1} because $\beta \in (0,1)$.

Therefore, instead of maximising the indirect utility, the government can maximise second-period income I^i_{t+1}, which in turn will maximise utility. Second-period income is: $w_{t+1}h(\tau^i_t B_t) + (1-\tau^i_t)b^i_t R_{t+1}$ where $w_{t+1} = w(k_{t+1}$ and $R_{t+1} = R(k_{t+1}$. At the same time, $k_{t+1} = \frac{(1-\tau_t)B_t}{h(\tau_t B_t)} = \frac{(1-\tau_t)B_t}{1+\frac{\tau_t B_t}{1+\tau_t B_t}}$. Putting everything together,

$$
\begin{aligned}
\tau^i_t &= \arg\max w_{t+1}h(\tau^i_t B_t) + (1-\tau^i_t)b^i_t R_{t+1} \\
&= \arg\max (1-\tau^i_t)^\alpha h(\tau^i_t B_t)^{1-\alpha} B^\alpha_t \left(1 - \alpha + \alpha\frac{b^i_t}{B_t}\right).
\end{aligned}
$$

Maximising with respect to τ^i_t yields:

$$\frac{\partial}{\partial \tau_t^i} = 0 \implies$$

$$B_t^\alpha \left(1 - \alpha + \alpha \frac{b_t^i}{B_t}\right) \left[\alpha(1-\tau_t^i)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha}(-1) + \right.$$
$$\left. +(1-\alpha)h'(\tau_t^i B_t)h(\tau_t^i B_t)^{-\alpha} B_t (1-\tau_t^i)^\alpha\right] = 0$$

$$\alpha(1-\tau)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} =$$
$$= (1-\alpha)h'(\tau_t^i B_t) B_t (1-\tau)^\alpha h(\tau_t^i B_t)^\alpha$$

$$\alpha(1-\tau)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} B_t^{\alpha-1} =$$
$$= (1-\alpha)h'(\tau_t^i B_t) B_t^\alpha (1-\tau)^\alpha h(\tau_t^i B_t)^\alpha$$

$$\underbrace{\alpha(1-\tau)^{\alpha-1} h(\tau_t^i B_t)^{1-\alpha} B_t^{\alpha-1}}_{R(k_{1+1})} =$$
$$= \underbrace{(1-\alpha) B_t^\alpha (1-\tau)^\alpha h(\tau_t^i B_t)^\alpha}_{w(k_{t+1})} h'(\tau_t^i B_t)$$

$$R(k_{t+1}) = w(k_{t+1}) h'(\tau_t^i B_t)$$

The optimal condition for τ_t^i does not involve the bequest received b_t^i. Consequently, everybody will agree on the optimality of the tax rate and it will be implemented. In our case, substituting and solving for τ_t:

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad B_t > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad B_t \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

Alternatively, it is possible to re-express the condition for positive taxation in terms of k_{t+1} :

$$\tau_t = \begin{cases} \frac{-B_t(1+2\alpha)+\sqrt{B_t^2(1+4(1+2B_t)(1-\alpha)\alpha)}}{4B_t^2\alpha} & \text{if} \quad k_{t+1} > \frac{\alpha}{1-\alpha} \\ 0 & \text{if} \quad k_{t+1} \leq \frac{\alpha}{1-\alpha} \end{cases} = \tau(B_t)$$

5.3 One economy, two groups

We suppose now that the economy, at time $t = 0$, consists of two groups: capitalists (C) and workers (W). The share that capitalists represent is denoted by λ_t. However, since all individuals always have one child, shares remain constant, this is, $\lambda_t = \lambda$. The unique difference between the two groups is the initial endowment of capital: * Capitalists own the initial stock of capital (which we assume is sufficiently large as to be able to bequest). * Workers have no capital, and thus give no bequests.

Therefore, the total amount of bequests in a given period is:

$$B_t = \lambda b_t^C + (1 - \lambda) b_t^W.$$

The remainder of the model is the same as before, in particular,

$$k_{t+1} = \frac{(1 - \tau(B_t))B_t}{h(\tau(B_t)B_t)}.$$

Such an economy shifts from having to classes of people to only one, where everybody owns capital. The critical transition occurs because of, eventually, capitalists find it optimal to impose a tax on themselves to finance public education. With it, and as wages keep increasing, workers are eventually able to give bequests, thus becoming capitalists. Instead of detailing the exact process (check the reference), we will simulate the economy for a set of parameters.

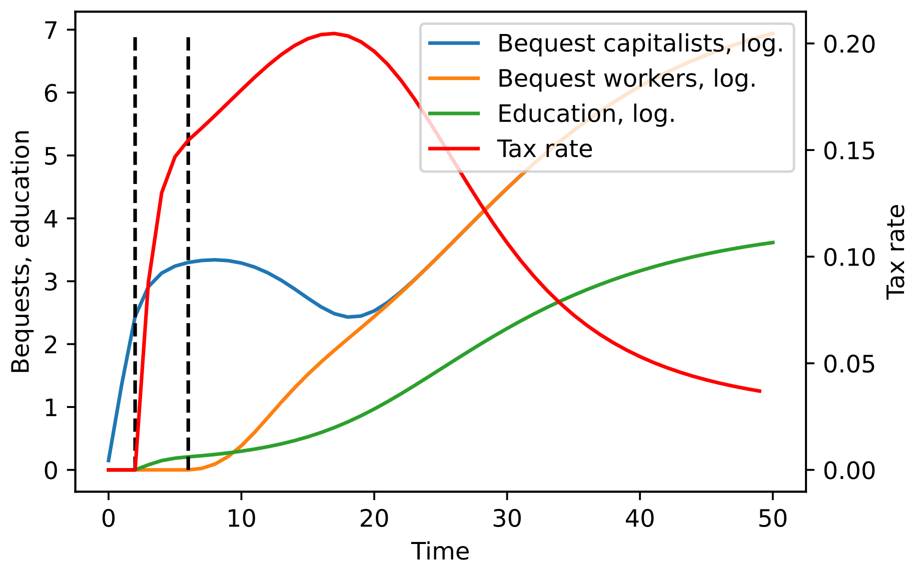


Figure 5.2: Simulated path of the economy proposed in Galor and Moav, 2006

6 Galor and Özak (2016)

Galor and Özak (2016) show empirically how differences in time preference (how much people discount the future) has an agricultural origin. This is important for development, because those who save more are the individuals who are more patient, this is, who are more future-oriented. Although the major contribution of the paper is empirical, the authors develop an OLG model from which they derive implications that are tested.

The theoretical model shows how the composition of a population can be modeled using the OLG framework. The most influential model in that regard is Bisin and Verdier (2001). However, the model in Galor and Özak is relatively simple and illustrates well some population dynamics.

6.1 The model

We work under the OLG framework, and we assume that the economy is agricultural and at the very early stages of development. In every period, the economy consists of individuals who live for **three** periods.

- During the first period of life, individuals are children and are economically passive. Consumption during this period is provided by parents.
- In the second period and third periods, individuals work
- All individuals can choose between two modes of production:

- Endowment mode: it provides an equal pay-off during the second and third periods of live. For instance, individuals may be hunters.
- Investment mode: it pays little during the second period of live, but the pay-off during the third period is much larger. This represents farmers: they must seed and wait for crops to grow.

Lastly, a crucial assumption of the mode is the **lack of financial markets and long-term storage technology**. This implies that individuals **cannot** transfers consumption between periods two and three. Hence, production in the second period has to be consumed in the second period; and consumption in the third period must be consumed in the third period.

6.1.1 Production modes

All adults must decide between the *endowment mode* or the *investment mode*. The endowment mode provides a constant level of output,$R^0 > 1$ in each of the working periods. If investment is instead chosen, it requires an investment during the first working period, which implies that less resources are available for consumption. In particular, we assume that it leaves individuals with 1 unit of consumption during their second period of life. However, the output it yields during the third period R^1 is higher than under *endowment*: $\ln(R^1) > 2\ln(R^0)$.

Finally, depending on the chosen production mode, the income of individual i is given by:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if endowment} \\ (1, R^1) & \text{if investment} \end{cases}$$

6.2 Preferences

A key aspect of the model to generate dynamics in the evolution of individual traits is the fertility decision. This approach is common: typically, fertility is linked to a trait through income. This is, individuals with more income will be able to have more children. If the trait is transmitted from parents to children, then the trait that allows generating more income will become more and more prevalent in the economy.[1] In any case, preferences are really important in this type of models, because fertility decisions are derived from them.

Every period t, a generation of size L_t becomes economically active, this is, reaches the second period of life. Those individuals were born in period $t-1$. At this stage, each individual will live for two periods. Remember that financial markets do not exist, and also that it is **impossible** to transfer resources between periods by storing them.

We assume that, during the second period of life, individuals only consume what they produce. Lastly, during the third and last period individuals consume and also have children. In particular, utility is given by:

$$u^{i,t} = \ln c_{i,t} + \beta_t^i[\gamma \ln n_{i,t+1} + (1-\gamma)\ln c_{i,t+1}], \quad \gamma \in (0,1),$$

where $c_{i,t}$ and $c_{i,t+1}$ are the levels of consumption in the second and third periods of life and $n_{i,t}$ is the number of children. It is important to comment on $\beta_t^i \in (0,1]$: it represents *individual* i's discount factor, this is, how much he values the future with respect to the present. The larger β_t^i, the more the value of future and, hence, the more patient the individual is. Notice that β_t^i changes over time and also by individual.

During the second period, individuals do not really make any decision: since resources cannot be transferred, all production must be consumed.

[1] Bisin and Verdier also model the case when parental indoctrination and how it affects the evolution of cultural traits.

Hence, $c_{i,t} = y_{i,t}$. However, during the last period individuals can trade-off utility from consumption and utility from children. The paper assumes that each child costs τ units of consumption, which gives rise the last-period budget constraint:

$$y_{i,t+1} = c_{i,t+1} + \tau n_{i,t+1}.$$

Considering the preferences, utility maximisation implies:

$$c_{i,t+1} = (1-\gamma) y_{i,t+1},$$

$$n_{i,t+1} = \frac{\gamma}{\tau} y_{i,t+1}.$$

Lastly, the indirect utility $(v_{i,t})$ of individual i is given by:

$$v_{i,t} = \ln y_{i,t} + \beta_t^i [\ln y_{i,t+1} + \xi], \quad \xi \equiv \gamma \ln\left(\frac{\gamma}{\tau}\right) + (1-\gamma)\ln(1-\gamma).$$

6.3 Hunters or farmers

Since individuals can decide on their mode of production, they are free to choose to become either hunters or farmers. This is, each individual will decide the mode of production (endowment or investment) that maximises lifetime utility. Hence,:

$$v_{i,t} = \begin{cases} \ln R^0 + \beta_t^i(\ln(R^0) + \xi) & \text{if endowment} \\ \ln 1 + \beta_t^i(\ln(R^1) + \xi) & \text{if investment} \end{cases}.$$

An individual is indifferent between modes of production if he obtains the same utility from both. This is, the individual with $\beta_t^i = \hat{\beta}$ is indifferent between becoming a hunter or a farmer if and only if:

$$\ln R^0 + \hat{\beta}(\ln(R^0) + \xi) = \ln 1 + \hat{\beta}(\ln(R^1) + \xi).$$

Solving for $\hat{\beta}$ allows us to identify such individual:[2]

$$\hat{\beta} = \frac{\ln R^0}{\ln R^1 - \ln R^0} \in (0, 1).$$

Lastly, all individuals with $\beta_t^i < \hat{\beta}$ will optimally choose the endowment technology while those with $\beta_t^i > \hat{\beta}$ find the investment technology optimal. Note that, as the return to agriculture increases (R^1 increases), the cutoff value $\hat{\beta}$ decreases:

$$\frac{\partial \hat{\beta}}{\partial R^1} = \frac{-\ln R^0}{R^1(\ln R^1 - \ln R^0)^2} < 0,$$

this is, as agriculture becomes more and more profitable, more individuals will find it optimal to become hunters.

Hence, we can rewrite the income of an individual as a function of his β_t^i:

$$(y_{i,t}, y_{i,t+1}) = \begin{cases} (R^0, R^0) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ (1, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Of course, since income in the last period of life is different, hunters and farmers will have different number of children. In particular, using the optimal number of children derived above:

$$n_{i,t+1} = \frac{\gamma}{\tau} y_{i,t+1} = \begin{cases} \frac{\gamma}{\tau} R^0 \equiv n^E & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \frac{\gamma}{\tau} R^1 \equiv n^I & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases}.$$

Because $R^1 > R^0$, farmers have more children than hunters.

[2] The assumption $\ln(R^1) > 2\ln(R^0)$ is important to establish that $\hat{\beta} \in (0, 1)$.

6.4 The evolution of preferences

Finally, we can compute how preferences change over time due to the differential fertility between farmers and hunters. This is, because farmers have more children than hunters, if we assume that preferences about time β_t^i are transmitted between parents and children, the share of farmers will increase over time. The paper assumes almost that, although modifies slightly the transmission of preferences for individuals engaging in farming. In particular: * β_t^i is perfectly transmitted if an individual is a hunter. * Farmers transmit a larger value of β_t^i to their children, reflecting an acquired tolerance to waiting and delaying reward. This is:

$$\beta_{i,t+1}^i = \begin{cases} \beta_t^i & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} \leq \hat{} \\ \phi(\beta_t^i, R^1) & \text{if } {}^{\mathrm{i}}_{\mathrm{t}} > \hat{} \end{cases},$$

with

- $\beta_t^i \leq \phi(\beta_t^i) < 1$: the transmitted β is always more than the one the parent had,
- $\phi(\hat{\beta}, R^1) > \beta_t^i$,
- $\phi_\beta(\beta_t^i, R^1) > 0$: the higher the value of β_{t+1}^i, the more it increases,
- $\phi_{\beta\beta}(\beta_t^i, R^1) < 0$: but at a decreasing rate,
- $\phi_R(\beta_t^i, R^1) > 0$: the higher the value of R, the more β_{t+1}^i increases.

Suppose an individual at the beginning of time who has $\beta_0^i < \hat{\beta}$. This individual will optimally decide to be a hunter, and according to the process for the transmission of preferences, his sons will inherit $\beta_1^i = \beta_0^i$. Because $\hat{\beta}$ is constant over time, all sons will decide to be hunters as well and transmit the same time preferences, over and over again. Hence, if $\beta_0^i \leq \hat{\beta} \implies \lim_{t\to\infty} \beta_t^i = \beta_0^i$.

Suppose instead that $\beta_0^i > \hat{\beta}$. The individual will become a farmer and transmit $\phi(\beta_0^i, R^1) > \beta_0^i$. Accordingly, all his sons will also be farmers and keep transmitting an ever-increasing value of β^i_{t+1}. However, because $\phi_{\beta\beta}(\beta^i_t, R^1) < 0$, the transmission process has a steady-state, this is, $\lim_{t\to\infty} \beta^i_t = \bar{\beta}^I$. Notice that $\bar{\beta}^I$ is the maximum level β^i_t can reach.

6.4.1 Proof (not in the paper)

We want to show that $\beta^i_{t+1} = \phi(\beta^i_t, R^1)$ has a unique steady-state. This amounts to showing that $\bar{\beta}^I = \phi(\bar{\beta}^I, R^1)$ for a unique value $\bar{\beta}^I$. Define $G(\beta) = \phi(\beta, R) - \beta$. Since we focus on farmers, we know that the very initial one in the dynasty had $\beta^i_0 > \hat{\beta}$. So, for our purposes, the function G has as domain $\beta \in [\hat{\beta}, \infty)$. We know that $G(\hat{\beta}) = \phi(\hat{\beta}, R^1) - \hat{\beta} > 0$ because $\phi(\hat{\beta}, R^1) > \beta^i_t$. Moreover, the function G has unique maximum at $\phi_\beta = 1$ because $\phi_{\beta\beta} < 0$, and the maximum is positive because it must be larger than $\phi(\hat{\beta}, R^1) - \hat{\beta} > 0$. After the maxima, the function continuously decreases, thus crossing only one the horizontal axis, this is, there is a unique value $\bar{\beta}$ such that $G(\bar{\beta}^I) = 0$, which constitutes the unique steady state.

6.5 Evolution of traits over time

Lastly, suppose that initially, at time $t = 0$, the initial population presents different levels of time preference. We assume that, initially, traits are characterised by some distribution of $\eta(\beta^i_0)$ with support $[0, \bar{\beta}^I]$. Furthermore, we normalise the initial generation to be of size one: $L_0 = 1$. Alternatively:

$$L_0 = \int_0^{\bar{\beta}^I} \eta(\beta^i_0) \mathrm{d}\beta^i_0 = 1.$$

We also know that all individuals whose $\beta_0^i \leq \hat{\beta}$ decide to use the endowment technology, and the remaining opt for the investment technology. Therefore, the size of hunters (E) and farmers (I) is given by:

$$L_0^E = \int_0^{\hat{\beta}} \eta(\beta_0^i)\mathrm{d}\beta_0^i,$$

$$L_0^I = \int_{\hat{\beta}}^{\bar{\beta}^I} \eta(\beta_0^i)\mathrm{d}\beta_0^i.$$

The number of individuals evolves according to the fertility rate of each group, this is,

$$L_t^E = L_0^E {n^E}^t = (\frac{\gamma}{\tau}R^0)^t L_0^E$$

$$L_t^I = L_0^I {n^I}^t = (\frac{\gamma}{\tau}R^1)^t L_0^I$$

and total population is $L_t = L_t^E + L_t^I$.

Finally, notice that the distribution of β_t^i does not change for those individuals with $\beta_t^i \leq \bar{\beta}$: in fact, they all have the same number of children, and each child inherits the trait of his parent. Therefore, the average value $\bar{\beta}^E$ is constant over time. In contrast, the average value $\bar{\beta}_t^I$ increases. In any case, at any given period t, the overall average value for time preference is given by:

$$\bar{\beta}_t = \theta_t^E \bar{\beta}_t^E + (1 - \theta_t^E)\bar{\beta}_t^I,$$

where θ_t^E is the fraction of individuals who engage in the endowment production process, and $\bar{\beta}_t$ is just the weighted average.

$$\theta_t^E = \frac{L_t^E}{L_t^E + L_t^I} = \frac{{R^0}^t}{{R^0}^t + {R^1}^t \frac{L_0^I}{L_0^E}}.$$

Hence, as time advances, the share of the population engaged in the endowment production process shrinks towards zero:

$$\lim_{t\to\infty} \theta_t^E = 0.$$

This process reflects their lower reproductive success.

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